Twoples 10/26.

Project ideas...

1. models of n-categories.
2. ∞-cats vs. model cats.
   - underlying ∞-cats
   - Quillen eq's $\Rightarrow$ equiv. of ∞-cats.
   - Dwyer-Kan localization.
3. (n,r)-categories & weak|strict n-categories.

N_Δ or N or N^{hc}

$$\text{Model cats} \xrightarrow{L^{DK}} sCat \xrightarrow{(-)^f} sCat \xrightarrow{N_\Delta} qCat \subseteq sSets$$

$$C \longmapsto L^{DK}C \longmapsto N_\Delta L^{DK} C =: C^\infty$$

- D-K localization or "simplicial" loc'n
- Hammock loc'n L^H

- "simplicial nerve"
- homotopy coherent nerve

CW-approx: $Top \to Top$, $X \mapsto X^f / X^c$ (??)

$(-)^f : \mathcal{M} \to \mathcal{M}$, $X \mapsto X^f$

fibrant replacement

X is fibrant if the map $X \xrightarrow{!} 1$ to the final object is a fibration

from axioms of model cats, can factor $X \to 1$...

$$X \xrightarrow{\sim} X^f \twoheadrightarrow 1.$$

We call the object RX the "fibrant replacement" of X.

eg. CW-complexes are (co?) fibrant replacement in $Top_{Quillen}$

eg. ∞-cats are fibrant objects in $sSet_{Joyal}$

eg. Kan cpxes are fibrant objects in $sSet_{Quillen}$

- Kerodon Ch. 3.
- "Quasicats & Kan cpxes" - Joyal

Kan cpxes $\subseteq$ qCat
∞-gpds

$Top_{Quillen} \underset{Q}{\simeq} sSet_{Quillen}$
$\cup$ CW $\longleftrightarrow$ Kan $\cup$

$X \in qCat$ (Kan $\subseteq$)

$hX \in Cat$, $\in Gpd$

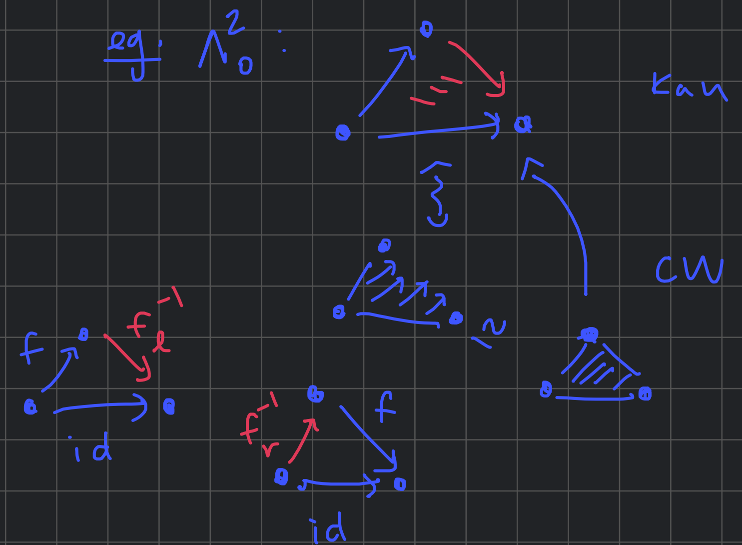
Kan: $(0 \le i \le n)$ all horns

$$\Lambda^n_i \to X, \quad \Lambda^n_i \hookrightarrow \Delta^n, \quad \Delta^n \dashrightarrow X$$

qCat: $(0 < i < n)$ inner horns

outer horn filling $\leftrightsquigarrow$ inverting maps.

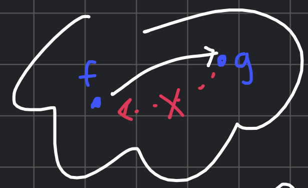
eg. Λ^2_0:



Why $sCat_{\text{Bergner}} \xrightarrow{(-)^f} sCat_{\text{Berg.}}$?

A priori N_Δ doesn't form ∞-cats for any simplicially enriched cat.

$\mathcal{C} \in Cat$ $\left(\begin{array}{l} ob \\ \{\mathcal{C}(x,y) \in set\} \end{array} \right.$ $\quad \mathcal{C} \in sCat$ $\left(\begin{array}{l} ob(\mathcal{C}) \\ \{\mathcal{C}(x,y)_\bullet \in sset\} \end{array} \right.$ $\longmapsto N_\Delta \mathcal{C} \in sSet$ $\left(\begin{array}{l} N_\Delta \mathcal{C}_0 = ob(\mathcal{C}) \\ N_\Delta \mathcal{C}_1 = \mathcal{C}(x,y)_0 \\ \vdots \end{array} \right.$



sSet $\mathcal{C}(x,y)_\bullet$

$x \underset{g}{\overset{f}{\rightrightarrows}} y$

If $f \sim g$, then there's an edge $h: f \to g$ in $\mathcal{C}(x,y)_1$

But ∞-cats shouldn't be arb. simp. cats...
they should have mapping spaces / Kan cpx instead of mapping ssets.

(homotopies ie. 2-morphisms in the $(\infty,1)$-cat)

Kan

So instead of considering all sCats... we look at only Kan-enriched cats. $\quad$ Kan-Cat $\subseteq$ sCat.

another model of $(\infty,1)$-cats.

Like how Kan cpxes were fibrant in sSet

Kan-Cat are fibrant objs in sCat.

Indeed, this is why $sCat_{\text{Bergner}}$ is interesting, because the $(\infty,1)$-cats are the fibrant objects.

So... $\quad$ sCat $\xrightarrow{(-)^f}$ Kan-Cat $\xrightarrow{N_\Delta}$ sSet, $\quad$ sCat $\to$ qCat, $\quad$ qCat $\subseteq$ sSet, $\quad$ Kan-Cat $\subseteq$ sCat

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The homotopy theory of ∞-cats
ie. want ∞-cat of ∞-cats..

$\underline{\underline{Cat_\infty}}$ ... $\quad$ Take qCat $\subseteq$ sSets.
This turns out to be a s-cat. Given $A, B \in sSet$, there's a sset $Fun(A,B)_\bullet \in sSet$

$\left\{ \begin{array}{l} 0: \ f: A \to B \\ 1: \ h: A \times \Delta^1 \to B \\ 2: \ \varphi: A \times \Delta^2 \to B \end{array} \right\}$

small ∞-cats ??? I think

When $B \in qCat$ I think $Fun(A,B) \in qCat$

Take $Fun(A,B)^\simeq \subseteq Fun(A,B)$

Kan cpx / ∞-gpd / Space $\qquad$ ∞-cats

$Cat_\infty^\Delta \in sCat$

One forms a Kan-enriched cat. of q cats using this

& then $N_\Delta(Cat_\infty^\Delta) =: Cat_\infty \in qCat$.

homotopy theories $\longleftrightarrow$ ∞-categories

• Cat_∞

model categories.

underlying + (project idea)

"Universal homotopy theories"

Q: homotopy theory of homotopy theories.

Working inside of a cat whose objects are homotopy theories.

model cats

∞-cats.

Given homotopy theories...

e.g. $\mathrm{Top}_{\mathrm{Quillen}} \underset{Q}{\simeq} \mathrm{sSet}_{\mathrm{Quillen}}$ the htpy theory of spaces / ssets

vs. qCat. Q: What is a "htpy equivalence" btwn ∞-categories?

Want to construct an ∞-cat Cat_∞ — ob = ∞-cats.

On the way to build Cat_∞, you need · N_Δ

· sCats

homotopical data!

To describe the htpy thy of ∞-cats...

...as a ∞-cat Cat_∞.

...as a model cat: the "Joyal model structure" on sSet. (∞-cats are the fibrant)

~~homotopy categories~~

qcats /

"anima" $\longleftrightarrow$ topos?

"anima" $\longleftrightarrow$ ∞-cat?